

HARER STABILITY AND ORBIFOLD COHOMOLOGY

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ABSTRACT. In this paper we review the combinatorics of the twisted sectors of $\mathcal{M}_{g,n}$, and we exhibit a formula for the age of each of them in terms of the combinatorial data. Then we show that orbifold cohomology of $\mathcal{M}_{g,n}$ when $g \rightarrow \infty$ reduces to its ordinary cohomology. We do this by showing that the twisted sector with minimal age is always the hyperelliptic twisted sector with all markings in the Weierstrass points; the age of the latter moduli space is just half its codimension in $\mathcal{M}_{g,n}$.

1. INTRODUCTION

In recent years there have been lots of new results on geometrical and topological properties of the moduli space $\mathcal{M}_{g,n}$ parametrizing smooth curves of genus g with n distinct marked points on it. This moduli space is a smooth Deligne–Mumford stack, or an orbifold, when $2g - 2 + n > 0$, and its coarse moduli space is a quasiprojective variety of dimension $3g - 3 + n$. When $n > 2g + 2$, every marked curve is rigid, therefore the moduli space is actually a smooth quasiprojective variety.

A celebrated result states that there are stabilization isomorphisms

$$(1) \quad H^k(\mathcal{M}_{g,n}, \mathbb{Q}) \rightarrow H^k(\mathcal{M}_{g+1,n}, \mathbb{Q}) \quad \text{when } 3k + 2 \leq 2g.$$

This isomorphisms are originally due to Harer in [Ha], but the ranges of their validity have been gradually improved over time by the efforts of different authors. This allows the definition of *stable cohomology*. The tautological classes κ_i are preserved by the above stabilization when g is sufficiently large. A recent result proven in [MW] by Madsen–Weiss shows then that the resulting map

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \rightarrow H^*(\mathcal{M}_\infty)$$

is also an isomorphism. We refer the reader to [Ek], [Ki] and [Wa] and to the references therein for more details about these topological results.

In the latest years, building on earlier results in topology [Ka] and theoretical physics [DHVW1], [DHVW2], it has become clearer that when studying the geometry and topology of orbifolds, one should include in the study the *twisted sectors* of the orbifold itself. We refer to [ALR] for an introduction to this emerging new subject. In particular, the cohomology theory of an orbifold is enriched by the so-called *orbifold cohomology*, introduced by Chen and Ruan in [CR]. As a graded vector space, the orbifold cohomology is the direct sum of the cohomology of the original orbifold and of the cohomology of the twisted sectors; the degree of the cohomology classes of the twisted sector is shifted by (twice) a rational number called *age* in the

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orbifold cohomology. This number is not of topological nature, in fact it depends on the complex structure. Its geometric significance appears in [JKK] as the virtual rank of an element in the rational K -theory of the twisted sector also known as “half of the normal bundle”, this element plays a key role in orbifold intersection theory.

In this note, we introduce the twisted sectors of $\mathcal{M}_{g,n}$ in the combinatorial description of [P], [PT], we write a closed formula for the age of the twisted sectors of $\mathcal{M}_{g,n}$ (this formula was written in [P] for $\mathcal{M}_{2,n}$ and in [PT] for \mathcal{M}_g). Our main result is Theorem 1, which states that for fixed (g, n) , the twisted sector with minimal age is the hyperelliptic twisted sector with marked Weierstrass points. Note that it is a well-known and classical fact that the twisted sectors of $\mathcal{M}_{g,n}$ have codimension higher than $g - 2 + n$, with equality only for the hyperelliptic locus. Our novel contribution here is that the virtual rank of “half of the normal bundle” (see above) is strictly greater than $\frac{g-2+n}{2}$, with equality only for the hyperelliptic twisted sector. To the best of our knowledge, the geometric significance of this numerical discovery is still mysterious.

Combining Theorem 1 with Harer stability, we obtain that the orbifold cohomology of $\mathcal{M}_{g,n}$ stabilizes. Combining further our main result with the theorem of Madsen–Weiss, we can explicitly compute the orbifold cohomology of $\mathcal{M}_{g,n}$ in low degrees. Indeed, from Theorem 1, we deduce

$$(2) \quad H_{orb}^k(\mathcal{M}_{g,n}, \mathbb{Q}) = H^k(\mathcal{M}_{g,n}, \mathbb{Q}), \quad \text{when } k < g - 2 + n.$$

Note that this equality is trivially true when $n > 2g + 2$ since there are no twisted sectors of $\mathcal{M}_{g,n}$ in that range.

The stabilization of orbifold cohomology was conjectured by Fantechi in the discussion following her talk [Fa] at MSRI. We acknowledge Fantechi for her insight in this topic. The author was supported by the DFG project Hu 337/6-2.

2. THE TWISTED SECTORS OF $\mathcal{M}_{g,n}$ AND THEIR AGE

In this section we review the combinatorics of the twisted sectors of $\mathcal{M}_{g,n}$. This description of the twisted sectors of $\mathcal{M}_{g,n}$ was obtained in [P] for $n \geq 1$ or $g = 2$, and in [PT] for the remaining cases $\mathcal{M}_{g,0}$, $g \geq 3$.

Let us fix (g, n) with $g > 0$ and $n \geq 1$ if $g = 1$. A (g, n) -admissible datum consists of non-negative integers $(g', N; d_1, \dots, d_{N-1}, a_1, \dots, a_{N-1})$, with $N \geq 2$ and such that

$$(3) \quad 2g - 2 = N(2g' - 2) + \sum_{i=1}^{N-1} (N - \gcd(i, N))d_i,$$

$$(4) \quad \sum_{i=1}^{N-1} i d_i \equiv 0 \pmod{N},$$

$$(5) \quad \sum_{i=1}^{N-1} a_i = n, \quad a_i \leq d_i, \quad a_i = 0 \text{ if } \gcd(i, N) \neq 1.$$

$$(6) \quad n = g' = 0 \implies \text{the g.c.d. of } N \text{ and of the } i\text{'s such that } d_i \neq 0 \text{ is } 1.$$

Each (g, n) -admissible datum corresponds to $\binom{n}{a_1, \dots, a_{N-1}}$ twisted sectors of $\mathcal{M}_{g,n}$ that are related each to the other by an (a_1, \dots, a_{N-1}) -permutation of the marked points. Since we will only investigate properties of the twisted sectors of $\mathcal{M}_{g,n}$ that do not depend on this permutation, from now on we shall slightly abuse the notation and identify each twisted sector Y of $\mathcal{M}_{g,n}$ with its (g, n) -admissible datum

$$Y \sim (g', N; d_1, \dots, d_{N-1}, a_1, \dots, a_{N-1}).$$

These facts follow from [P, Proposition 2.13] for $n \geq 1$ and from [PT, Corollary 2.15, Theorem 2.18] in the case $n = 0$.

For completeness, we briefly recall our description of the twisted sectors of $\mathcal{M}_{g,n}$, from which the above correspondence follows. For more details, we refer to [P, Section 2.b] for the case $n \geq 1$ and to [PT, Section 2.b] for the case $n = 0$.

Construction 1. A twisted sector of $\mathcal{M}_{g,n}$ parametrizes connected cyclic covers of order N of curves of genus g' with total space a curve of genus g , where the n marked points are chosen among the points of total ramification. The branch divisor of the cyclic cover splits into $N-1$ divisors, some of which are possibly empty. Indeed to any point p in the branch divisor D , let q be any point in the fiber of p under the cyclic cover map; we define H_p as the stabilizer of the action of \mathbb{Z}_N at q , and ψ_p as the character of the action of H_p on the cotangent space in q . Then, for $0 < i < N$, we define D_i as the subset of D of those p such that $H_p = \langle i \rangle < \mathbb{Z}_N$, and such that¹ $\psi_p(i) = \omega_N$. In addition to g, g', N , the admissible datum consists of $d_i := |D_i|$ and of a_i , the number of chosen marked points in the preimage of D_i under the cyclic cover map.

Given a (g, n) -admissible datum, we can construct a moduli space of cyclic covers as in the paragraph above. Condition (4) is a compatibility condition that guarantees the existence of a (not necessarily connected) cyclic cover with the data d_i and N , condition (5) corresponds to the fact that the marked points must be points of total ramification for the cover. Condition (3) is Riemann–Hurwitz formula saying that the total space of the cover has genus g . Now if $n \geq 1$, it is easy to see that the total space of the cover is forced to be connected and that the moduli space parametrizing such covers is also connected. If instead $n = 0$, it is shown in [PT, Theorem 2.18] that there is always one connected component of the moduli space that parametrizes *connected* cyclic covers. This component may possibly be empty only when $g' = 0$, condition (6) rules out precisely these cases.

Given a twisted sector $(g', N; d_1, \dots, d_{N-1}, a_1, \dots, a_{N-1})$, its *codimension* in $\mathcal{M}_{g,n}$ is

$$(7) \quad \text{codim}(Y) := 3g - 3g' - \sum_{i=1}^{N-1} d_i + n,$$

its *twin* is $(g', N; d_{N-1}, \dots, d_1, a_{N-1}, \dots, a_1)$. If (g, n) is fixed and $n \leq 2g+2$, the *hyperelliptic twisted sector with n marked Weierstrass points* is $(g' = 0, N = 2; d_1 = 2g+2, a_1 = n)$. In short, we will also call it simply the

¹here ω_N is the canonical choice of a generator for the N -th root of 1 over \mathbb{C}

hyperelliptic twisted sector, from (7) it has codimension $g - 2 + n$. We now review the following well-known fact.

Lemma 1. *The codimension of any twisted sector Y of $\mathcal{M}_{g,n}$ satisfies $\text{codim}(Y) \geq g - 2 + n$, with equality if and only if Y is the hyperelliptic twisted sector with n marked Weierstrass points.*

Proof. Our statement is reduced to proving the inequality:

$$(8) \quad \sum d_i \leq 2g - 3g' + 2.$$

Using Formula (3), we have that:

$$(9) \quad \frac{N}{2} \sum d_i \leq \sum d_i (N - \gcd(i, N)) = 2g - 2 - N(2g' - 2),$$

therefore, it is enough to show that:

$$\frac{2}{N} (2g - 2 - N(2g' - 2)) \leq 2g - 3g' + 2.$$

Or, rearranging the terms, that:

$$(10) \quad (2N - 4)(g - 1) + Ng' \geq 0.$$

This is clearly always true. Equality holds if and only if g' equals 0 and N equals 2. \square

Every twisted sector Y is assigned a rational number, first defined by Chen–Ruan in [CR], which is called *degree shifting number*, *age*, or *fermionic shift*. Orbifold cohomology is then the direct sum of the ordinary cohomology and of the cohomology of all the twisted sectors, where the latter is shifted in degree by twice the age.

The age of a twisted sector of $\mathcal{M}_{g,n}$ can explicitly be determined in terms of its admissible datum. From [PT, Proposition 5.6] and [P, Lemma 4.5], we have the following formula for the age:

$$(11) \quad a(Y) = \frac{(3g' - 3)(N - 1)}{2} + \frac{1}{N} \sum_{\gcd(i, N)=1} a_i \sum_{k=1}^{N-1} k \sigma(k, i) + \\ + \frac{1}{N} \sum_{i=1}^{N-1} d_i \sum_{k=1}^{N-1} k \left(\left\{ \frac{ki}{N} \right\} + \sigma(k, i) \right),$$

where

$$\sigma(k, i) := \begin{cases} 0 & ki + \gcd(i, N) \equiv 0 \pmod{N} \\ 1 & ki + \gcd(i, N) \not\equiv 0 \pmod{N}. \end{cases}$$

Using only² (11), it is an easy exercise to check that, if Y and Y' are twins, the following holds:

$$(12) \quad a(Y) + a(Y') = \text{codim}(Y) = \text{codim}(Y').$$

For completeness, we briefly review the Chen–Ruan definition of degree shifting number, building on Construction 1.

²this can be obtained from the general definition of age, which we will give in (13).

Construction 2. Let $f : Y \rightarrow \mathcal{M}_{g,n}$ be the natural map from the twisted sector to the moduli stack of curves. The group μ_N acts on $f^*(T_{\mathcal{M}_{g,n}})$, the action can be diagonalized, and each eigenvalue at a point of Y has the form $\lambda_k = e^{2\pi i \alpha_k}$, where the $\alpha_k \in [0, 1[\cap \mathbb{Q}$ are the “logarithms of the eigenvalues”. It is not difficult to see that the function $\sum_k \alpha_k$ is (well defined) and constant on Y , thus we can define the *age* of Y to be

$$(13) \quad a(Y) := \sum_k \alpha_k \in \mathbb{Q}.$$

Moreover, by the very definition of twisted sector, the action of μ_N on T_Y is trivial, thus in the definition (13) it is equivalent to sum the “logarithms of the eigenvalues” of the normal bundle $N_Y \mathcal{M}_{g,n}$, where the latter is defined by the exact sequence of vector bundles

$$0 \rightarrow T_Y \rightarrow f^*(T_{\mathcal{M}_{g,n}}) \rightarrow N_Y \mathcal{M}_{g,n} \rightarrow 0.$$

The age of a twisted sector can be interpreted as the virtual rank of an element in the rational K -theory of Y that plays an important role in orbifold intersection theory, see [JKK, Definition 1.3, Sections 1.3 and 4].

3. THE TWISTED SECTORS OF MINIMAL AGE

Using only the combinatorial description of the previous section, and in analogy with Lemma 1, we can prove the main result of this note.

Theorem 1. *The age of any twisted sector Y satisfies $2a(Y) \geq g - 2 + n$, with equality if and only if Y is the hyperelliptic twisted sector with n marked Weierstrass points.*

From this, the following corollary relevant for orbifold cohomology follows:

$$(14) \quad H^k(\mathcal{M}_{g,n}) = H_{orb}^k(\mathcal{M}_{g,n}) \quad \text{if } k < g - 2 + n.$$

This is a consequence of the previous theorem when $n \leq 2g + 2$. When $n > 2g + 2$ the moduli stack $\mathcal{M}_{g,n}$ is actually a scheme, and then its orbifold cohomology equals its ordinary cohomology by definition. Note also that our bound on the cohomological degree k is sharp.

Using the stability results for ordinary cohomology (1), we deduce:

Corollary 1. *If $g > 4 - 3n$, then the isomorphisms (1) induce isomorphisms*

$$H_{orb}^k(\mathcal{M}_{g,n}, \mathbb{Q}) \rightarrow H_{orb}^k(\mathcal{M}_{g+1,n}, \mathbb{Q}) \quad \text{when } 3k + 2 \leq 2g.$$

In the remaining cases, the above isomorphisms hold iff $k < g - 2 + n$.

Note that the remaining cases occur only when $g = 0$ (when orbifold and ordinary cohomology coincide) and in the cases $(1, 1), (2, 0), (3, 0), (4, 0)$. In these special cases, our ranges for k are clearly optimal. In the other cases, our ranges coincide with the ranges of stability for ordinary cohomology (cf. (1)). These ranges are known to be optimal in the cases when $g \equiv 2 \pmod 3$. More details on the sharpness of the ranges for ordinary cohomological stability can be found in [Wa, p.2].

Corollary 2. *Orbifold cohomology of $\mathcal{M}_{g,n}$ stabilizes when $g \rightarrow \infty$ and the stable orbifold cohomology of $\mathcal{M}_{g,n}$ coincides with its ordinary stable cohomology.*

We now move to the proof of Theorem 1. Thanks to Lemma 1 and to (12), what we have to prove is in fact

$$(15) \quad |a(Y) - a(Y')| \leq \text{codim}(Y) - (g - 2 + n) = 2g + 2 - 3g' - \sum d_i,$$

with equality only when Y is the hyperelliptic twisted sector with n marked Weierstrass points. We introduce some notation. Let Σ be the set of proper divisors of N

$$\Sigma := \{d \in \mathbb{N} \mid d \text{ divides } N, d \neq N\},$$

and let

$$(16) \quad a(Y)_{\text{mark}} := \frac{1}{N} \sum_{\gcd(i, N)=1} a_i \sum_{k=1}^{N-1} k \sigma(k, i),$$

$$(17) \quad a(Y)_\sigma := \frac{1}{N} \sum_{\gcd(i, N)=\sigma} d_i \sum_{k=1}^{N-1} k \left(\left\{ \frac{ki}{N} \right\} + \sigma(k, i) \right)$$

We can rewrite our formula (11) for the age of a twisted sector Y as:

$$a(Y) = \frac{(3g' - 3)(N - 1)}{2} + a(Y)_{\text{mark}} + \sum_{\sigma \in \Sigma} a(Y)_\sigma.$$

The term $a(Y)_{\text{mark}}$ is the contribution to the age of Y coming from the marked points, and as such it is zero when $n = 0$. Of course now we have the estimate

$$(18) \quad |a(Y) - a(Y')| \leq |a(Y)_{\text{mark}} - a(Y')_{\text{mark}}| + \sum_{\sigma \in \Sigma} |a(Y)_\sigma - a(Y')_\sigma|.$$

We can give estimates for each term in the right hand side of (18).

Lemma 2. *The following inequalities hold:*

$$(19) \quad |a(Y)_{\text{mark}} - a(Y')_{\text{mark}}| \leq \frac{N-2}{N} \sum_{\gcd(i, N)=1} a_i,$$

$$(20) \quad |a(Y)_\sigma - a(Y')_\sigma| \leq \frac{(N-2\sigma)(\frac{N}{\sigma} + 5)}{6N} \sum_{\gcd(i, N)=\sigma} d_i.$$

Proof. Let us begin with the contribution coming from the marked points. The left hand side of (19) is equal to

$$(21) \quad \frac{1}{N} \left| \sum a_i (\lambda(i) - \lambda(N-i)) \right|,$$

where $\lambda(s)$ is the multiplicative inverse of s modulo N . The maximum of the absolute value of

$$\lambda(i) - \lambda(N-i) = 2\lambda(i) - N,$$

is obtained when i is either 1 or $N-1$.

As for the second inequality, we separate the two addends in the right hand side of (17). For the first term, consider the function of i

$$g_N^\sigma(i) := \left| \sum_{k=1}^{N-1} k \left(\left\{ \frac{ik}{N} \right\} - \left\{ \frac{(N-i)k}{N} \right\} \right) \right|.$$

Its maximum among the values of i such that $\gcd(i, N) = \sigma$ is obtained for $i = \sigma$ or for $i = N - \sigma$. From this, we obtain

$$(22) \quad \left| \sum_{k=1}^{N-1} k \left\{ \frac{ik}{N} \right\} - \left\{ \frac{(N-i)k}{N} \right\} \right| \leq g_N^\sigma(\sigma) = \frac{1}{6} \left(\frac{N}{\sigma} - 1 \right) (N - 2\sigma).$$

The second term is treated similarly to the contribution coming from the marked points. The maximum of the absolute value of

$$\sum_k (\sigma(k, i) - \sigma(k, N - i)) = 2i - N$$

is obtained when i is either σ or $N - \sigma$. Combining this fact with (22), we get the desired inequality. \square

Proof. (of Theorem 1) As we have already observed, it suffices to prove (15). By using the Riemann-Hurwitz formula (3) to eliminate the term g , the right hand side of (15) can be rearranged to:

$$(2N - 3)g' - 2(N - 2) + \sum_{\sigma \in \Sigma} (N - \sigma - 1) \sum_{\gcd(i, N) = \sigma} d_i.$$

Now let us define for convenience the function:

$$f_N(\sigma) := (N - \sigma - 1) - \frac{(N - 2\sigma)(\frac{N}{\sigma} + 5)}{6N} = \frac{(6 - \frac{1}{\sigma}) N^2 - (6\sigma + 9)N + 10\sigma}{6N},$$

for any integer $N \geq 2$ and any σ a real number between 1 and $N/2$. By using (19) and (20), in order to prove (15) it is enough to prove:

$$(23) \quad -\frac{N-2}{N} \sum_{\gcd(i, N)=1} a_i + \sum_{\sigma \in \Sigma} f_N(\sigma) \sum_{\gcd(i, N)=\sigma} d_i \geq (2 - 2g')(N - 2) - g',$$

with equality only in the case of the hyperelliptic twisted sector. Note that $a_i \leq d_i$ from inequality (5).

The left hand side of (23) is always nonnegative for any integer $N \geq 2$, because

$$\tilde{f}_N(1) := f_N(1) - \frac{N-2}{N} = \frac{(N-2)(5N-11)}{6N} \geq 0.$$

Therefore when $g' > 0$, the strict inequality (23) holds evidently, as the right hand side is strictly smaller than 0. Thus all we have to prove is (23) when $g' = 0$, a case in which we always have that $\sum d_i \geq 3$ (this follows from (3) posing $g' = 0$ and $g > 0$).

We start with the case $g' = 0$ and $\sum d_i \geq 4$. The function f_N is concave, thus it has its minimum either in 1 or in $N/2$, and we have $f_N(N/2) = \frac{N-2}{2}$. The following two inequalities hold in this case

$$(24) \quad \tilde{f}_N(1) \sum d_i \leq 2(N - 2),$$

$$(25) \quad f_N(N/2) \sum d_i \leq 2(N - 2),$$

and they suffice to prove (23). If (23) is an equality, then either (24) or (25) must be an equality. If N equals 2, we are precisely in the case of the hyperelliptic twisted sector. If $N > 2$, the inequality (24) is strict, so (25) must be an equality and therefore $\sum d_i = 4$. So if (23) is an equality, with

$g' = 0, N > 2$ and $\sum d_i = 4$, then $d_{\frac{N}{2}} = 4$, but this implies $g = 1$ by (3), hence $n \geq 1$, and this case does not exist because of (5).

So we are left with the case $g' = 0$ and $\sum d_i = 3$. Most of the twisted sectors still fall into this last category, but not the hyperelliptic twisted sector. We pose the three nonzero d_i 's to be $d_{\sigma_1} = d_{\sigma_2} = d_{\sigma_3} = 1$. Then it suffices to prove the strict inequality

$$(26) \quad \left(6 - \sum_{i=1}^3 \frac{1}{\sigma_i}\right) N^2 - \left(3 + 6 \sum_{i=1}^3 \sigma_i\right) N + 10 \sum_{i=1}^3 \sigma_i > 6n(N - 2).$$

If N is fixed, there are only finitely many possibilities for the variables involved in (26). The constraints are:

$$(27) \quad \begin{cases} \sigma_1 + \sigma_2 + \sigma_3 < N, \\ a\sigma_1 + b\sigma_2 + c\sigma_3 = N \quad \text{for some } a, b, c \in \mathbb{N}^+, \\ n \leq |\{i \mid \sigma_i = 1\}| \leq 3, \\ \sigma_i \text{ divides } N, \sigma_i \neq N, \end{cases}$$

where all the quantities involved are integers. The first is a consequence of Riemann-Hurwitz (3) (assuming $g > 1$), the second follows from (4) and the third from (5). From now on, we aim at proving (26) for N greater than a certain explicit constant. We will repeatedly use that the left hand side of (26), for fixed n, N , is a concave function in the domain of definition (27).

We can also assume for convenience that $\sigma_1 \leq \sigma_2 \leq \sigma_3$.

- If $n = 3$, then from (27) we deduce $\sigma_1 = \sigma_2 = \sigma_3 = 1$. The inequality (26) is satisfied when $N > 11$.
- If $n = 2$, from (27), we have that $\sigma_1 = \sigma_2 = 1$. It is enough to check (26) for the extreme values $\sigma_3 = 1$ and $\sigma_3 = N/2$. The first follows from the case $n = 3$, by checking the case of $\sigma_3 = N/2$ we see that (26) is valid when $N > 22$.
- If $n = 1$, from (27) $\sigma_1 = 1$, so we have $1 \leq \sigma_2 \leq \sigma_3 \leq N/2$ and $\sigma_2 + \sigma_3 < N - 1$. It is enough to check the extremal values. The case when $\sigma_2 = 1$ follows from the case $n = 2$. From the second point in (27), if $\sigma_3 = N/2$, then σ_2 is either 1 or 2; in the latter case (26) is valid when $N > 14$. Finally, when $\sigma_2 = \sigma_3 = N/3$, (26) is always valid.
- If $n = 0$, we can assume $\sigma_i \geq 2$, since the other cases fall in the above paragraph. Moreover, there are six extremal cases that fulfill the first and the last of (27):

$$(2, 2, 2), (2, 2, \frac{N}{2}), (2, \frac{N}{3}, \frac{N}{2}), (\frac{N}{7}, \frac{N}{3}, \frac{N}{2}), (\frac{N}{5}, \frac{N}{4}, \frac{N}{2}), (\frac{N}{4}, \frac{N}{3}, \frac{N}{3}).$$

We check that (26) for the extremal cases is satisfied when $N > 36$ (the inequality is sharp in the case of the fourth triple).

To conclude the proof, we have to check that (15) holds in the cases when $g' = 0$, $\sum d_i = 3$ and $N < 37$, which imply $g < 18$. These cases are only finitely many, and can be handled with the help of a computer program³. \square

³The source code of a C++ program that lists all twisted sectors of \mathcal{M}_g , each one with its age, is available at <http://www.iag.uni-hannover.de/~pagani/twisted.cpp>.

To conclude, we remark that condition (6) has not been used in any of the steps of the proof of Theorem 1, which could then have been stated slightly more generally for the twisted sectors of the moduli spaces of not necessarily connected smooth curves of genus g .

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